Some hints for the homework questions: (November 2002)

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2. part gimel:

The idea is to find a natural way to identify V with $\mathbb{R}^3$.

Look for a map which sends elements ("points") of V to points of $\mathbb{R}^3$.

This map should be one-to-one, and it should set up a natural connection between the inner product in V with the usual inner product in $\mathbb{R}^3$. This should mean that the map also sets up a natural connection between the norm in V and the usual norm in $\mathbb{R}^3$.

Now apply this map to *all* the "points" in W. This will give you a set of points in $\mathbb{R}^3$. It should not be too too hard to describe this new set. Then you will know the answer to 2 gimel.

Remark: This exercise is intended to help you have more insight into a very powerful and beautiful idea which is used in Fourier series and in several other parts of mathematics. The idea is that the intuition and knowledge which we have naturally about geometry and points in three dimensional space (because we live in it!) can give us clues about what to do in other spaces, usually of higher dimensions, where the "points" in these spaces are functions or matrices or other mathematical objects that we need to study.

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3. alef. This follows from standard properties of integrals and limits that you should know from hedva 1 or hedva 2.

3. bet. Let f be continuous on $[0, \infty)$. For each positive integer n, let I_n be the integral of $|f|$ on the interval $[n-1, n]$. Find a condition on the sequence of numbers $\{I_n\}_{n \in \mathbb{N}}$ which is equivalent to the condition $f \in V$. Find a particular sequence of numbers $\{I_n\}$ which satisfies this condition. Find a non negative continuous function f whose integral on $[n-1, n]$ equals I_n (from the numbers that you just chose) and such that $f(n - 1/2) > n$.

3. gimel. Use standard estimates.

3. dalet. The question asks us to consider $f = \mu S$, where S is an infinite sum of characteristic functions of intervals, times numbers α_n.

At what points x can it perhaps happen that S is not continuous? What property of μ guarantees that f is continous despite possible discontinuities of S . Similar ideas to those in the hint for part alef should help you show that W is contained in V.

Now just follow the definitions. Here is one fact that can be useful: Suppose that $f = \mu S$, as above. Then, for each $n > 0$, let f_n be the function which is zero on $[0, 2\pi n)$ and equals f on $[2\pi n, \infty)$. Show that $f_n \in V$. Find a formula for $||f_n||$. What happens to $||f_n||$ as n tends to ∞? Here are some related questions, which you should know how to solve from hedva 1 or hedva 2. Let $\{b_n\}$ be a sequence of non-negative numbers.

If the series $\sum_{n=1}^{\infty} b_n$ converges, does this imply that $\sum_{n=1}^{\infty} b_n^2$ converges?

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Here are some other questions which might help you somewhere in dalet.

Find a non zero function $g \in V$ such that $g(x) = 0$ for all $x > 2\pi$, and g is orthogonal to μ. (There are many different ways to do this. It is perhaps a bit easier in the case when μ is a non negative function. Then you can, for example, let $g(x) = a \sin x$ on $[0, \pi]$ and $g(x) = b \sin x$ on $[\pi, 2\pi]$ for suitable numbers a and b.

Oops!! Actually we have to be a bit more careful. What if μ vanishes on all the interval $[0, \pi]$ or all the interval $[\pi, 2\pi]$. Then this kind of function g will not be orthogonal to μ. But then it is not hard to fix this problem. How?.

After you treat this case it is easy to use it to treat the general case.)

Then find the function in W which is nearest to this function g.

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5. First here are some things that you should know from Hedva. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of real (or complex !!) numbers. If $\lim_{n\to\infty} b_n$ exists does this imply that $\lim_{n\to\infty} b_{2n}$ exists? Does it imply that $\lim_{n\to\infty} b_{2n-1}$ exists? If one of these latter two limits exist does this imply that $\lim_{n\to\infty} b_n$ exists? What if both of them exist?

Now suppose that $b_n = \sum_{m=1}^{n} a_m$ for some other sequence $\{a_n\}_{n\in\mathbb{N}}$ which satisfies $\lim_{n\to\infty} a_n = 0$. Can it happen that the three sequences $\{b_n\}$, $\{b_{2n}\}$ and $\{b_{2n-1}\}$ converge to different limits? Or can some of them not have a limit when others do have a limit?

These results do not apply automatically to question 5, but some easy generalizations of them to the case where a_n and b_n are vectors in a normed space do apply.

Now let us look more explicitly at parts alef and bet:

alef: Does the limit $\lim_{n\to\infty} \langle f, v_n\rangle$ exist? Does the limit $\lim_{n\to\infty} \|\langle f, v_n\rangle v_n\|$ exist?

bet: Does the limit $\lim_{n\to\infty} \langle f, v_n\rangle v_n(x)$ exist for some (or all?) constant points $x \in [a, b]$?